ALGORITHMS

Algorithms

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The *K-12 Mathematics Standards* were approved by the State Board of Education in 2008 (*K-8 Mathematics Standards* on April 28 and *9-12 Mathematics Standards* on July 30). These standards specify not only what students at each grade and in each high school course need to learn but also what teachers need to teach. The Washington Assessment of Student Learning (WASL) for grades 3-8 will align with these standards in spring 2010, and the 10th-grade WASL and new end-of-course tests for some of the high school courses should be ready in spring 2011. (There will be more information available about these tests by late fall 2008.)

One of the major differences between the Grade Level Expectations (GLEs) and the *K-12 Mathematics Standards* is the explicit mention of fluency with algorithms, especially in grades 3-8. The introductory text for the standards describes what is meant by fluency:

> "The term fluency is used in these standards to describe the expected level and depth of a student's knowledge of a computational procedure. For the purposes of these standards, a student is considered fluent when the procedure can be performed immediately and accurately. Also, when fluent, the student knows when it is appropriate to use a particular procedure in a problem or situation. A student who is fluent in a procedure has a tool that can be applied reflexively and doesn't distract from the task of solving the problem at hand. The procedure is stored in long-term memory, leaving working memory available to focus on the problem."

Fluency with algorithms sets the stage for more sophisticated understanding of mathematics ideas. For example, fluency with the long division algorithm can be a basis for understanding the repeating decimal representation of rational numbers as well as for understanding division of polynomials. In 1998, the National Council of Teachers of Mathematics published a yearbook focused on algorithms (Morrow & Kenney, 1998). The chapters in that book provide information about the history of algorithms within mathematics as well as suggestions for how to teach algorithms to students.

What is an algorithm? How do you know when algorithms are the same or different?

In a simple way, an algorithm can be thought of as a set of step-by-step actions that, if carried out correctly, always produces a correct answer. Stephen B. Maurer, a Ph.D. mathematician, provided a more detailed description:

> "An algorithm is a precise, systematic method for solving a class of problems. An algorithm takes *input*, follows a *determinate* set of rules, and in a *finite* number of steps gives *output* that provides a *conclusive* answer. *Determinate* means that for each allowed input, the first action is precisely specified and, more generally, that after each action in the sequence the next action is precisely specified. *Conclusive* usually means that the output correctly solves the problem, but it can mean that the algorithm either solves the problem correctly or announces that it cannot solve it." (Maurer, 1998, p. 21, emphasis in the original)

A critical part of this description is that the algorithm solves a class of problems. There would be little reason to develop an algorithm that applies to only one (or a few) situations.

Maurer goes further to note that there may be multiple ways for recording an algorithm. Specifically, he provides an example (pp. 21-22) for how the product, 432 x 378, is typically written by people who can multiply fluently:

$$\begin{array}{r} 432 \\ \underline{\times\, 378} \\ 3456 \\ 3024 \\ \underline{1296} \\ 163296 \end{array}$$

He notes that “to see the basic one-digit-by-one-digit steps, expand the first intermediate row to show more detail:

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    4 3 2
x       8
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      1 6
    2 4
  3 2
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  3 4 5 6
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Of course, the calculation is not usually written this way. To save space, the ‘carries’ are all either done mentally or marked with small digits” (p. 22).

One point here is that there is a difference between the algorithm itself and the scheme for recording that algorithm. The “partial products” expansion of part of the computation still represents the same algorithm; it is merely recorded differently. At the level of specifying what to write down, the algorithms might be thought of as different, since the directions (i.e., steps) for what symbols to write down would be different. However, in terms of the mathematical basis of the algorithm, the two recording schemes (i.e., the short form versus the longer, partial-products form) are the same algorithm. The standards allow for the possibility of different recording schemes (i.e., the Explanatory Comment that appears with Performance Expectations 3.1.C, 4.1.F, and 5.1.C): “Teachers should be aware that in some countries the algorithms might be recorded differently.”

Why are algorithms important?

Algorithms are, and always have been, an important part of mathematics. Indeed, the search for algorithms has spurred research in mathematics for centuries, and development of new algorithms continues today. For example, Steven Skiena in the Department of Computer Science at Stony Brook University maintains The Stony Brook Algorithm Repository (http://www.cs.sunysb.edu/~algorith/). These algorithms are sorted into categories such as combinatorial problems, computational geometry, and set and string problems. It is no accident that this repository should be maintained by a computer scientist; a computer program is, after all, a set of instructions that takes an input and generates a specific output; that is, a computer program is an algorithm. Computer implementation of algorithms is obviously a critical part of today’s world.

Algorithms for the four operations on whole numbers and rational numbers were important for the development of commerce, when bookkeeping was all done on paper. But compass and straight-edge constructions, included as part of the *9-12 Mathematics Standards* and originally developed centuries ago, are also algorithms. And some of us are old enough to have learned how to compute a square root with a paper-and-pencil algorithm! Algorithms are also found in other sciences. For example, staining cells (e.g. the Gram stain for better microscopic clarity), rapid HIV detection, and manufacture of specific compounds such as sulfuric acid all require the use of algorithms. Algorithms as tools for finding solutions to real world problems are abundant and pervasive in our culture; they are so common we hardly notice how much we rely on them. Students need to appreciate the place of algorithms in both history and current applications.

What is the role of algorithms in the mathematics standards?

Because algorithms have been, and continue to be, important in understanding mathematics, they are addressed explicitly in the *K-12 Mathematics Standards*. Equally important, however, is the need for students to develop fluency with common algorithms so that they have a base on which to build more sophisticated mathematical understanding. Instruction on algorithms cannot start with the short, standard versions; few students are ready to learn immediately at this very abstract level. Rather, instruction needs to begin with more transparent computation procedures that can ultimately be generalized to the standard versions.

Campbell, Rowan, and Suarez (1998) cite three critical attributes of student-invented algorithms that actually apply to all algorithms, whether student-invented or not: efficient, mathematically valid and generalizable. The standard algorithms meet these criteria, but the “transitional algorithms” that teachers use to help students progress toward fluency with the standard algorithms also need to meet these criteria. Efficiency is described as “efficient enough to be used regularly without considerable loss of time and without frustration due to the number of recorded steps required” (p. 51). For example, the “partial products algorithm for multiplication” presented above might be more efficient early in the study of multiplication in the sense that it might cause less frustration for students in keeping track of what the “small digits” (Maurer, 1998, p. 22) signify in the short form of the multiplication algorithm. The remainder of this paper is a brief discussion of the role of algorithms in each of the grade bands: K-2, 3-5, 6-8, and 9-12.

Algorithms in grades K-2

One major focus for grades K-2 is to develop a deep sense of number, number relations and foundations for operations. In grades K-1, students focus on deeply understanding whole number quantities and combinations to 5, 10 and 20. Students count sets of objects, and use numerals to label sets and to label and represent quantities. Students put quantities together (composing), take quantities apart (decomposing) and lay a foundation for the actions (operations) of joining (addition) and separating (subtraction).

In grade 2 students solidify their understanding of place value and develop procedures for adding and subtracting two-digit numbers efficiently and accurately. “Procedures” are not quite as formal as “algorithms;” for example, to subtract 17 from 53, students might

1. subtract 10 from 53 to get 43,
2. realize that 7 is greater than the 3 in 43, so decompose 7 as 3 + 4,
3. subtract 3 from 43 to get 40, and
4. subtract 4 from 40 to get 36 (or count backwards from 40 to 36).

This procedure has elements that lead to the standard subtraction algorithm, but the order and organization of the steps is different. However, this procedure is accurate and for many students it is efficient. Further, it is consistent with a “mental math” approach to subtracting that even some adults might use. Students who develop this procedure have a strong foundation for learning the standard algorithm in grade 3. As noted by Fosnot and Dolk (2001), computation instruction does not begin with formal or standard algorithms.

> “The starting point of any computation work needs to be children’s own constructions. To teach any strategy, including algorithms, directly by focusing on the procedures will only cause children to adopt the procedures and stop thinking. But teachers do have a role in ensuring that efficient strategies are being developed [and] in ensuring that children are not left with only their initial inventions…” (p. 123)

Algorithms in grades 3-5

One major focus for grades 3-5 is fluent and accurate use of standard algorithms for addition, subtraction, multiplication and division of whole numbers. The word “standard” is first used in Performance Expectation 3.1.C to describe the algorithms students need to understand and be able to use. “Standard” refers to algorithms that have been commonly and widely used in the United States. Similar to learning procedures for addition and subtraction in second grade, instruction leading to use of standard regrouping algorithms for addition and subtraction begins with more informal and transparent forms. During instruction, explicit connections need to be made between less formal algorithms and the standard algorithms. Primarily, this is done by helping students connect understanding of place value to understanding of the conceptual basis for each operation. In grade 3 students learn to use standard regrouping algorithms for addition and subtraction; in grade 4 students learn to use the standard multiplication algorithm; and in grade 5 students learn to use the standard long division algorithm. Throughout, students are expected to understand the connections between addition and subtraction and between multiplication and division.

Fractions and decimals are another focus in grades 3-5. In grades 3 and 4, students learn to represent fractions and decimals and to relate fractions and decimals. In grade 5 students learn to add and subtract decimals and prepare for study of multiplication and division of fractions and decimals in grade 6.

It is important to note that the performance expectations for grades 3-5 do not say students will only use the standard algorithms, nor do they say students will use standard algorithms every time they add, subtract, multiply or divide. Part of the power of using standard/ written algorithms is knowing when to use them and when to use some other strategy. Students need to develop a repertoire for computation that includes both standard algorithms and mental math strategies that build on number sense, number relations and properties of operations.

Algorithms in grades 6-8

Development of algorithms continues into grades 6, 7 and 8. In sixth grade, students learn to multiply and divide fractions and formalize the order of operations (a set of rules that specifies what order to complete computations). For example, students apply order of operations directly to simplify expressions involving more than one operation. An indirect application of order of operations is solving a linear equation by “undoing,” or working backwards through the order of operations to isolate the unknown. Although this procedure is not a formal algorithm, this skill builds on fluent use of order of operations. By grade 8, students should be fluent with all four operations on all rational numbers (including integers and negative rational

numbers) and apply these skills in situations such as computing the mean of a set of data.

Formulas, such as those for surface area and volume of a variety of two- and three- dimensional figures, can also be thought of as algorithms; they are procedures to arrive at an answer. Indeed, many of these formulas can be found preprogrammed, and all can be programmed, in scientific calculators. In grades 7 and 8, students encounter linear equations and the concept of slope. If the coordinates of two points on a line are given, there is a formula for finding the slope of the line; that formula can be thought of as an algorithm. (Have you ever heard, "Slope is rise over run"? This is just a shortened version of a formula/ algorithm.) Other mathematical ideas such as unit conversions, prime factorization, ratios, proportional reasoning and scatter plots involve algorithmic aspects. Indeed, much of the mathematics in grades 6, 7 and 8 is about becoming proficient, efficient and knowledgeable with a wide spectrum of algorithmic processes upon which higher mathematics will be launched.

Algorithms in high school

When a student enters high school with a strong arithmetic background – including fluency with standard algorithms and deep understanding of fractions, decimals, ratios and proportions -- instruction can address the subtleties and nuances of algebra and higher mathematics. This deep understanding is part of the mathematics knowledge and reasoning that will serve students throughout their lives. "[R]esearch shows that completion of Algebra II correlates significantly with success in college and earnings from employment" (National Mathematics Panel Final Report, 2008, p. xiii). In part, this economic advantage may be due to familiarity with the idea of algorithms as tools for accomplishing tasks. Knowing how to apply algorithmic tools appropriately is an important part of work in the 21st century.

Indeed, high school mathematics "feels different" than the mathematics of earlier grades. In Algebra I, for example, students not only learn algorithms, they begin to develop and justify the correctness of algorithms such as completing the square (i.e., Performance Expectation A1.5.D). They also begin to develop the idea of an algorithm as a "tool" and learn the roles algorithms can play, both within mathematics and in the real world. Every operation or function "button" on a calculator (e.g., the linear regression coefficient "button") has behind it an algorithm to transform an input into a correct output. In high school students learn to interpret the outputs of these buttons within appropriate mathematical limits, even when the specific algorithms behind those buttons might not be understandable until the tools of calculus and infinite series are developed in later mathematics courses.

Of course, just using algorithms accurately is not enough for the deep mathematical understanding our students need and deserve. Students need to develop deep understanding about the concepts and process that underlie algorithms. All students should leave high school with knowledge of, and appreciation for, both simple computation algorithms (e.g., division of fractions) and more abstract "tool algorithms" (e.g., using technology to model a best-fit line for a set of data). This progression to abstraction is a subtle and vital piece of the education of all students. Students should understand not only that behind every calculator button there is a mathematical algorithm that was created by people who understood the relevant mathematics but also that people will create new algorithms in the future. It would not be an understatement to say that much of our progress as humans is due to the development and use of algorithms. Try to imagine life without them!

References

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